Bestimmung der S-Matrix

Betrachte infinitesimale Lorentz Trafo:

Boost

$$\Omega^{xb} = \begin{pmatrix} \cosh\eta & -\sinh\eta & 0 & 0 \\ -\sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tanh\eta = \beta_x = \frac{v_x}{c}$$

Lorentzfaktor

$$\cosh\eta = \frac{1}{\sqrt{1-\tanh^2\eta}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\sinh\eta = \sqrt{\cosh^2\eta - 1} = \frac{\beta}{\sqrt{1-\beta^2}}$$

$$\rightarrow d\Omega^{xb} = \begin{pmatrix} 1 & -d\eta & 0 & 0 \\ d\eta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4 + d\eta \underbrace{\begin{pmatrix} 0 & -1 & & 0 \\ 1 & 0 & & \\ & & & \\ 0 & & & 0 \end{pmatrix}}_{J_{0x}}$$

Drehung

$$\Omega^{xd} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta & 0 \\ 0 & \sin\vartheta & \cos\vartheta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

„Generator der Drehung"

$$\rightarrow d\Omega^{xd} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -d\vartheta & 0 \\ 0 & d\vartheta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbb{1}_4 + d\vartheta_{xx} \underbrace{\begin{pmatrix} 0 & -1 & & 0 \\ 1 & 0 & & \\ & & & \\ 0 & & & 0 \end{pmatrix}}_{= J_{xx}}$$

→ Lorentz trafo immer „Drehung" in 4d-Raumzeit

Man kann zeigen, dass beliebige infinitesimale LT als folgendes geschrieben werden kann:

$$\Omega^{\alpha}{}_{\beta} = \mathbb{1} + \frac{i}{2}\theta_{\mu\nu}\left[J^{\mu\nu}\right]^{\alpha}{}_{\beta} \qquad \text{(in 4d-Raumzeit)}$$

$$\text{mit } \left[J^{\mu\nu}\right]^{\alpha}{}_{\beta} = i\left[g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta}\right] \text{ und } \theta_{\mu\nu} = -\theta_{\nu\mu}$$

↑ Generatoren Tensor (Matrix (4d) aus 4d-Generator-Matrizen)

↑ infinitesimaler Winkeltensor → Matrix voller $d\vartheta, d\eta$

Beide antisymmetrisch unter Vertauschung von ν, μ

Boosts: $K_n = J^{0n}$, $n \in \{1,2,3\}$; „echte" Rotation: $L_m = \frac{1}{2}\varepsilon_{mlk} J^{lk}$; $m,l,k \in \{1,2,3\}$

$$[L_n, L_m] = i\varepsilon_{nmk} L_k \quad ; \quad [K_n, K_m] = -i\varepsilon_{nmk} L_k \quad ; \quad [K_n, L_m] = -i\varepsilon_{nmk} L_k$$

Hieraus können die S-Matrizen folgendermaßen gefunden werden:

$$S(\Omega) = \mathbb{1} + \frac{i}{2}\theta_{\nu\mu}\tilde{J}^{\mu\nu} = \mathbb{1} - \frac{i}{2}\theta_{\nu\mu}\tilde{J}^{\nu\mu}, \text{ da } \tilde{J}^{\mu\nu} = \frac{1}{2}\sigma^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

Bsp: Boost in x-Richtung:

$$d\Omega^x = \mathbb{1} + \frac{i}{2}\theta_x J^{0x} \simeq \text{TR von } \exp\left(\frac{i}{2}\theta_x K^x\right)$$

$$K^x = i\left[g^{0\alpha}\delta^x{}_\beta - g^{x\alpha}\delta^0{}_\beta\right] = i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$S(\Omega^x) = \mathbb{1} + \frac{i}{2}\theta_x \tilde{J}^{0x} \simeq \text{TR von } \exp\frac{i}{2}\theta_x \tilde{J}^{0x}$$

$$\tilde{J}^{0x} = \frac{i}{4}[\gamma^0, \gamma^x] = \frac{i}{4}\left((\sigma_3 \otimes \sigma_0)(i\sigma_2 \otimes \sigma_x) - (i\sigma_2 \otimes \sigma_x)(\sigma_3 \otimes \sigma_0)\right) = \frac{1}{2}\sigma_x \otimes \sigma_x$$

$$= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \frac{1}{2} i \gamma^0 \gamma^x = \frac{1}{2} i \alpha^{\alpha}_{x}$$